

# Dissipative hydrodynamic models for the diffusion of impurities in a gas\*

Stephane Brull<sup>†</sup>

Lorenzo Pareschi<sup>‡</sup>

December 22, 2004

## Abstract

Recently linear dissipative models of the Boltzmann equation have been introduced in [7, 5]. In this work, we consider the problem of constructing suitable hydrodynamic approximations for such models.

## 1 Introduction

The dissipative linear Boltzmann equation describes the dynamic of a set of particles with mass  $m_1$  interacting inelastically with a background gas in thermodynamical equilibrium composed of particles with mass  $m \ll m_1$ . For example, the case of fine polluting impurities interacting with air or another gas is investigated in [4].

As observed in [7], the only conserved quantity is the number of inelastic particles and as a result, a conventional hydrodynamic approach of Euler type leads to a single equation describing the advection (or advection-diffusion at the Navier-Stokes order) of inelastic particles at the velocity of the background.

The aim of this note is to find hydrodynamic models for such Boltzmann equation which posses equations for the momentum and the temperature of the gas. Here, we present a

---

\*Support by the European network HYKE, funded by the EC as contract HPRN-CT-2002-00282, is acknowledged.

<sup>†</sup>LATP. Centre de mathématiques et d'informatique. Université de Provence, 39 rue Joliot-Curie, 13453 Marseille. France. Email: [brull@cmi.univ-mrs.fr](mailto:brull@cmi.univ-mrs.fr)

<sup>‡</sup>Department of Mathematics, University of Ferrara, Via Machiavelli 35, 44100 Ferrara, Italy. E-mail: [pareschi@dm.unife.it](mailto:pareschi@dm.unife.it)

closed set of dissipative Euler equations for a pseudo-Maxwellian case which generalizes the one considered in [7].

Let us mention that the problem of finding suitable hydrodynamics for inelastic interacting gases has been studied recently by several authors (see [1, 3, 2, 6] and the references therein).

The paper is organized as follows. Section 2 deals with the linear dissipative Boltzmann model and the pseudo-Maxwellian approximation. Section 3 is devoted to discuss the problem of the closure for the moment equations and to derive a dissipative Euler system.

## 2 The dissipative linear Boltzmann equation

We consider the dissipative linear Boltzmann equation

$$\frac{\partial f}{\partial t}(t, x, v) + v \cdot \nabla_x f(t, x, v) = Q(f)(t, x, v), \quad (2.1)$$

with,

$$\frac{1}{2\pi\lambda} \int_{\mathbb{R}_v^3 \times S^2} B(v, w, n) \left[ \frac{1}{e^2} f(v_*) M_1(w_*) - f(v) M_1(w) \right] dw dn. \quad (2.2)$$

Here,  $B(v, w, n)$  denotes the collision kernel,  $\lambda$  the mean free path and  $e$  the restitution coefficient with  $0 < e < 1$ . The case  $e = 1$  corresponds to the elastic collision mechanism.

For the hard spheres model, the particles are assumed to be ideally elastic balls and the corresponding collision kernel is given by

$$B(v, w, n) = |q|, \quad (2.3)$$

with  $q = v - w$ . The background is assumed to be in thermodynamic equilibrium with given mass velocity  $u_1$  and temperature  $T_1$  i.e. its distribution function  $M_1$  is the normalized Maxwellian given by

$$M_1(v) = \frac{\rho_1}{(2\pi T_1)^{\frac{3}{2}}} \exp \left( - \frac{(v - u_1)^2}{2T_1} \right). \quad (2.4)$$

Mass ratio and inelasticity are described by the following dimensionless parameters,

$$\alpha = \frac{m_1}{m_1 + m} \quad \text{and} \quad \beta = \frac{1 - e}{2}, \quad (2.5)$$

where  $0 < \alpha < 1$  and  $0 < \beta < \frac{1}{2}$ .

In these conditions, it's possible to prove (see [7, 5]) that the stationary equilibrium states of the collision operators are given by the Maxwellian distributions

$$M^\sharp(v) = \left(\frac{m}{2\pi T^\sharp}\right)^{3/2} \exp\left\{-\frac{m(v-u_1)^2}{2T^\sharp}\right\}, \quad v \in \mathbb{R}^3, \quad (2.6)$$

having the same mean velocity of the background and temperature

$$T^\sharp = \frac{(1-\alpha)(1-\beta)}{1-\alpha(1-\beta)} T_1 \quad (2.7)$$

lower than the background one.

Here by analogy with [1], we consider an approximation of the hard sphere model characterized by the assumption

$$|v-w| \simeq S(t, x), \quad (2.8)$$

where  $S(t, x)$  is a suitable function which takes into account the fact that we have large relaxation rates for  $|v-w|$  large and small relaxation rates for  $|v-w|$  small. Clearly since  $v$  is distributed accordingly to  $f$  and  $w$  accordingly to  $M_1$  the function  $S$  cannot be simply a function of the temperature of a single gas as in [1].

On the other hand,  $M_1$  is given by (2.4) and thus a possible choice here consists in taking the expected value for  $|v-w|$  as choice of  $S$ . This gives

$$S(x, t) = \int_{\mathbb{R}_v^3} \int_{\mathbb{R}_w^3} |v-w| f(x, v, t) M_1(x, w) dw dv = \int_{\mathbb{R}_v^3} Z(x, v) f(v) dv, \quad (2.9)$$

with

$$Z(x, v) = \int_{\mathbb{R}_w^3} |v-w| M_1(x, w) dw. \quad (2.10)$$

Of course simpler choices can be done. For example, similarly to the case of a single gas, taking  $S(x, t) = \mu \sqrt{T_r(x, t)}$  for a suitable constant  $\mu$ , where  $T_r$  is the normalized relative “temperature” given by

$$T_r(x, t) = \frac{1}{3\rho(x)} \int f(x, v, t) |v - u_1(x)|^2 dv.$$

Note that at variance with [1] here the “temperature”  $T_r$  of the inelastic gas is measured with respect to the mean velocity of the background. Thus only asymptotically for large times it will correspond to the physical temperature.

Therefore this pseudo-Maxwellian model is given by

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{S(t, x)}{2\pi\lambda} \int_{\mathbb{R}_v^3 \times S^2} \left[ \frac{1}{e^2} f(v_*) M_1(w_*) - f(v) M_1(w) \right] dw dn. \quad (2.11)$$

The above model represents a better approximation of the hard sphere model with respect to the Maxwellian model considered in [7] which corresponds simply to  $S(x, t) = \text{const}$ .

### 3 Hydrodynamic limit and the Euler equation.

To avoid the term  $\frac{1}{e^2}$  in (2.11), it is useful to consider the weak form of (2.11). More precisely, let us define with  $\langle \cdot, \cdot \rangle$  the inner product in  $L^1(\mathbb{R}^3)$ . Given any regular test-function  $\varphi(v)$ , it holds that

$$\langle \varphi, Q(f) \rangle = \frac{S(t, x)}{\lambda\pi} \int_{\mathbb{R}_w^3} \int_{\mathbb{R}_v^3 \times S^2} (\varphi(v^*) - \varphi(v)) f(v) M_1(w) dw dn, \quad (3.12)$$

where the post-collisional velocity  $v^*$  is defined by

$$v^* = v - 2\alpha(1 - \beta)(q \cdot n)n. \quad (3.13)$$

Clearly  $\varphi = 1$  is a collision invariant whereas  $\varphi = v$  and  $\varphi = v^2$  are not.

The existence of a Maxwellian equilibrium at non-zero temperature (2.6) allows to construct hydrodynamic models for the considered granular flow. However, here only the mass of the inelastic particles is preserved. Thus the mass  $\rho$  is the unique hydrodynamic variable and the Euler system is reduced to the single advection equation [7]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u_1) = 0. \quad (3.14)$$

At this point, in order to perform a closure for the moment equations such that the equations for the mean velocity and the temperature of particles are preserved we assume the distribution function  $f$  to be the local Maxwellian at the mean velocity and temperature of the gas

$$M(x, v, t) = \frac{\rho(x, t)}{(2\pi T(x, t))^{\frac{3}{2}}} \exp \left( - \frac{(v - u(x, v))^2}{2T(x, t)} \right). \quad (3.15)$$

Taking  $\varphi = v$  in (3.12) leads to

$$\langle v, Q(M) \rangle = \frac{-2\alpha(1 - \beta)S(t, x)}{\lambda\pi} \int_{\mathbb{R}_w^3} \int_{\mathbb{R}_v^3} M(v) M_1(w) \left( \int_{S^2} (q \cdot n) n dn \right) dw dv. \quad (3.16)$$

Following ([2],[5]), we get

$$\int_{S^2} (q \cdot n) n dn = \frac{4\pi}{3} q. \quad (3.17)$$

So, (3.16) has the following expression

$$\langle v, Q(M) \rangle = \frac{-8\pi\alpha(1-\beta)S(t,x)}{3\lambda} \int_{\mathbb{R}_w^3} \int_{\mathbb{R}_v^3} M(v)M_1(w)(v-w)dw dv. \quad (3.18)$$

As,

$$\int_{\mathbb{R}_v^3} v M(v) dv = \rho u \quad \text{and} \quad \int_{\mathbb{R}_v^3} w M_1(w) dv = \rho_1 u_1, \quad (3.19)$$

the first moment equation has the expression,

$$\frac{\partial}{\partial t} u + \nabla \cdot (\rho u \otimes u) + \nabla_x (\rho T) = \frac{-4S(t,x)\alpha(1-\beta)}{3\lambda} \rho \rho_1 (u - u_1). \quad (3.20)$$

For the second moment, let us compute (3.12) with  $\varphi = \frac{1}{2}|v|^2$ . Hence,

$$\begin{aligned} \langle \frac{1}{2}|v|^2, Q(M) \rangle &= \frac{S(t,x)}{\lambda\pi} \int_{\mathbb{R}_w^3} \int_{\mathbb{R}_v^3 \times S^2} [-2\alpha(1-\beta)(q \cdot n)(v \cdot n) \\ &\quad + 4\alpha^2(1-\beta)^2|q \cdot n|^2] M(v)M_1(w)dw dv dn. \end{aligned} \quad (3.21)$$

Reasoning as in ([2],[5]), it holds that

$$\int_{S^2} |q \cdot n|^2 dn = \frac{2\pi}{3} |q|^2, \quad (3.22)$$

$$\int_{S^2} (q \cdot n)(v \cdot n) dn = \frac{2\pi}{3} (q \cdot v). \quad (3.23)$$

So, integrating the right-hand side of (3.21) with respect to the  $n$  variable and using (3.22) leads to

$$\begin{aligned} \langle \frac{1}{2}|v|^2, Q(M) \rangle &= \frac{-4\alpha(1-\beta)S(t,x)}{3\lambda} \int_{\mathbb{R}_w^3} \int_{\mathbb{R}_v^3} (q \cdot v) M(v)M_1(w)dw dv \\ &\quad + \frac{4\alpha^2(1-\beta)^2S(t,x)}{3\lambda} \int_{\mathbb{R}_w^3} \int_{\mathbb{R}_v^3} |q|^2 M(v)M_1(w)dw dv. \end{aligned} \quad (3.24)$$

As,  $q = v - w$ ,

$$\int_{\mathbb{R}_w^3} \int_{\mathbb{R}_v^3} |q|^2 M(v) M_1(w) dw dv = \int_{\mathbb{R}_w^3} \int_{\mathbb{R}_v^3} (|v|^2 - 2v \cdot w + |w|^2) M(v) M_1(w) dw dv. \quad (3.25)$$

As,

$$\frac{1}{2} \int_{\mathbb{R}_v^3} |v|^2 M(v) dv = \rho \left( \frac{1}{2} |u|^2 + \frac{3}{2} T \right),$$

it follows that

$$\int_{\mathbb{R}_w^3} \int_{\mathbb{R}_v^3} |q|^2 M(v) M_1(w) dw dv = \rho \rho_1 (3T + 3T_1 + |u|^2 + |u_1|^2 - 2u_1 \cdot u). \quad (3.26)$$

and

$$\int_{\mathbb{R}_w^3} \int_{\mathbb{R}_v^3} (|v|^2 - v \cdot w) M(v) M_1(w) dw dv = \rho \rho_1 (3T + |u|^2 - u \cdot u_1). \quad (3.27)$$

By (3.26) and (3.27), the right-hand side of (3.24) is equal to

$$2\alpha^2(1 - \beta)^2 \rho \rho_1 (3T + 3T_1 + |u|^2 + |u_1|^2 - 2u_1 \cdot u) - 2\alpha(1 - \beta) \rho \rho_1 (3T + |u|^2 - u \cdot u_1). \quad (3.28)$$

Finally, the left-hand side of (3.24) being computed by (3.26), we find the following dissipative Euler system

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) &= 0, \\ \frac{\partial}{\partial t} u + \nabla \cdot (\rho u \otimes u) + \nabla_x (\rho T) &= \frac{4\pi S(t, x) \alpha (1 - \beta)}{3\lambda} \rho \rho_1 (u_1 - u) \quad (3.29) \\ \frac{\partial}{\partial t} \left( \rho u \left( \frac{1}{2} |u|^2 + \frac{3}{2} T \right) \right) + \nabla \cdot \left( \rho \left( \frac{1}{2} |u|^2 + \frac{5}{2} T \right) u \right) &= \frac{4\pi \rho \rho_1 S(t, x)}{3\lambda} D(x, t) \end{aligned}$$

where

$$D(x, t) = \alpha^2(1 - \beta)^2 (3T + 3T_1 + |u|^2 + |u_1|^2 - 2u_1 \cdot u) - \alpha(1 - \beta) (3T + |u|^2 - u \cdot u_1). \quad (3.30)$$

## 4 Conclusion

We derived hydrodynamic approximations for linear dissipative Boltzmann equations that keep the equations for the mean velocity and the temperature of particles. To this aim the closure of the moment system is performed with respect to a local Maxwellian state which is not an equilibrium state for the Boltzmann operator. In this way a dissipative Euler system is derived.

## References

- [1] A.V.Bobylev, J.A.Carrillo, I.M.Gamba. *On some properties of kinetic and hydrodynamic equations for inelastic interactions*, J. Stat. Phys.,98, 743 - 773, (2000).
- [2] M.Bisi, G.Spiga, G.Toscani. *Grad's equations and hydrodynamics for weakly inelastic granular flows*, Physics of fluid (2004).
- [3] D.Benedetto, E.Caglioti, F.Golse, M.Pulvirenti. *A hydrodynamic model arising in the context of granular media*. Comput. Math. Appl. 38 (1999), no. 7-8, 121–131.
- [4] V. Garzó, J. M. Montanero, *Diffusion of impurities in a granular gas*, Physical Review E, 69 (2004).
- [5] B.Loods, G.Toscani. *The dissipative linear Boltzmann equation for hard spheres*, J.Statist.Phys., (2004).
- [6] L.Pareschi, G.Toscani. *Modelling and numerical methods for granular gases*, Modelling and computational Methods for Kinetic Equation, P.Degon, L.Pareschi and Russo Eds, Birkhauser, Boston (2004) 259-285.
- [7] G.Spiga, G.Toscani. *The dissipative linear Boltzmann equation*, Appl. Math. Letters 17 (3) 255-301 (2004).